

Collective Fields for QCD

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Abstract

A gauge-symmetric approach to effective Lagrangians is described with special emphasis on derivations of effective low-energy Lagrangians from QCD. The examples we discuss are based on exact rewritings of cut-off QCD in terms of new collective degrees of freedom. These cut-off Lagrangians are thus “effective” in the sense that they explicitly contain some of the physical long-distance degrees of freedom from the outset.

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It is well known that a description of physics in terms of given field variables $\phi_i(x)$ and a corresponding action $S[\phi_i]$ is far from being unique. There are numerous examples in quantum field theory where exactly the same physical results can be obtained by entirely different actions $\tilde{S}[\phi_i]$ in terms of different variables $\tilde{\phi}_i$. This is a consequence of the *field redefinition theorem*, which under very specific conditions ensures that S -matrix elements remain unaltered under a change of basis of the fields (for a partial list of references, see, e.g., ref. [1]). From a modern path-integral point of view the field redefinition theorem may seem almost trivial since it appears to express nothing but the possibility of changing variables inside an integral, but there are in fact subtleties at the two-loop level [2, 3]. These stem from the fact that a path integral is, after all, *not* just an ordinary integral. But when these problems appear, and how they can be avoided, is by now well understood.

More surprising is the fact that even such a highly non-trivial example as (1+1)-dimensional *bosonization* [4] also can be viewed as a genuine field redefinition [5, 6]. This is most easily seen in the collective field formalism which is at the heart of the effective Lagrangian method we will describe here. This technique also reveals how bosonization (and fermionization) are only two extremes of a continuum of equivalent field theory descriptions that contain, in general, both bosons and fermions with non-trivial interactions. In this sense, the massive Thirring model and the Sine-Gordon model are both effective Lagrangian descriptions of the same physics. One is no more fundamental than the other, but they may each have different domains in which they are more simple – more “effective” – in describing physics. In fact, and this turns out to be of crucial importance, both can be viewed as particular *gauge fixings* of a “higher” gauge-symmetric theory that contains both bosons and fermions [6].

We are hereby implying that the concept of effective Lagrangians must be broadened to include also the description of the same theory in terms of more convenient variables. As an example, Lagrangians written in terms of local fields which we can associate with physical states should thus, in many instances, be far more “effective” than one described by means of underlying fundamental fields.

Knowing that the gauge-symmetric collective field technique [7, 5] can be used to understand two-dimensional bosonization (and actually much more besides) [6], it is an obvious challenge to try to extend the same method to a full-fledged four-dimensional field theory such as QCD. The extent to which this is possible has recently been discussed in ref. [8], and the purpose of this talk is to provide a short review of that work. We shall try to highlight both weak and strong points of such an approach. In the course of our investigation we came across work by Zaks and by Karchev and Slavnov [9] which implicitly contain related ideas. There are also clear connections to the more conventional approach toward a derivation of chiral Lagrangians from QCD [10]. Related considerations can be of relevance for the electroweak theory [11]; see also ref. [12].

We begin by explaining the idea that *any* quantum field theory described by an action $S[\phi_i]$ (and an associated measure in the functional integral) can be considered as the gauge-fixed version of a class of “higher” gauge-symmetric Lagrangians. To find such a higher theory is not difficult. One first performs a field-enlarging transformation $G_a : \phi_i \rightarrow \phi'_i$, or $\phi_i = g_i(\phi', a)$, where $a_\alpha(x)$ are what we will call collective fields. They can at this stage be considered as parametrizing the transformation G_a . Under this transformation the action should transform as a scalar, i.e. $S'[\phi', a] \equiv S[g_i(\phi', a)]$. We

will only consider (invertible) transformations connected to the identity, which we take to occur at $a_\alpha(x) = 0$ (and this can always be arranged). Under this transformation G_a , the measure will pick up a Jacobian $\det |M_{ij}(\phi', a)| \equiv \det |\delta g_i(\phi', a)/\delta \phi'_j(x)|$. We next integrate over the transformation fields $a_\alpha(x)$ in the functional integral. Since these fields can be removed completely from both action and measure (by performing the inverse transformation G_a^{-1} on ϕ'_i), integrating over $a_\alpha(x)$ does not affect any physics. It does, however, make the path integral rather ill-behaved due to the “volume” of the integration over $a_\alpha(x)$. In the equivalent formulation in which $a_\alpha(x)$ is part of both action and (new) measure, this manifests itself in the emergence of a new *local gauge symmetry*:

$$\delta \phi'_i(x) = - \int dy dz M_{ij}^{-1}(x, y) \frac{\delta g_j(\phi'(y), a(y))}{\delta a_\alpha(z)} \Lambda_\alpha(z) , \quad \delta a_\alpha(x) = \Lambda_\alpha(x) , \quad (1)$$

where $\Lambda_\alpha(x)$ is the gauge transformation parameter. When we gauge-fix on the surface $a_\alpha(x) = 0$, we simply recover the theory in the original formulation. Other gauges will in general correspond to field redefinitions. This means that we can trade some of the original fields $\phi_i(x)$ for some of the collective fields $a_\alpha(x)$.

The first question to answer is: Why do we choose to perform field redefinitions using what appears to be such an elaborate route? (First we extend the basis of fields by new collective fields, then we immediately gauge-fix the same number of excessive degrees of freedom away). The reason is that this is an extremely efficient way of performing field redefinitions of arbitrary complexity. One *can* always obtain the same redefinitions by direct changes of variables in the functional integral, but it is not obvious how to find the corresponding direct transformation. In contrast, using standard (BRST) gauge-fixing technology this is accomplished in one simple step. Furthermore, *we need not know* the precise form of the field-redefinition transformation; the change of variables can be done in an entirely indirect manner through the gauge fixing. But it is important to stress that the gauge symmetry (1) should *not* be given any physical meaning. It is an artifact of the method.

Let us now try to apply this technique to QCD. By *collective* excitations of QCD, we mean those colour-singlet bound states that can appear in asymptotic states (or at least have a sensible finite lifetime). These are the mesons, glueballs, baryons, “hybrids”, etc. The simplest collective fields of QCD are those describing the pseudoscalar bosons, and it is those we will focus on here. We will thus set out to find the closest we can get to “bosonizing” some of the fermionic QCD degrees of freedom. This we will do first in the (flavour) abelian case by trying to extract directly from the QCD Lagrangian a field with the quantum numbers of the η' meson.

Our starting point is a generating functional for QCD of the form

$$\mathcal{Z}_{QCD}[V, A] = \int \mathcal{D}[\bar{\psi}, \psi] \mathcal{D}\mu[G] e^{-\int d^4x \mathcal{L}_{QCD}} \\ \mathcal{L}_{QCD} = \bar{\psi}(x)(\not{\partial} - i\not{A}(x) - \not{M}(x) - i\not{A}(x)\gamma_5)\psi(x) + \frac{1}{4g^2} \text{tr} G_{\mu\nu}(x) G_{\mu\nu}(x) . \quad (2)$$

Here $V_\mu(x)$ is an external vector source and $A_\mu(x)$ an external axial vector source, both Abelian (diagonal in the $SU(N_f)$ flavour indices). The vector potential $G_\mu(x)$ is the usual gluon field, here for convenience generalized to $SU(N_c)$, and $G_{\mu\nu}(x)$ is the corresponding

field strength tensor. The usual $SU(N_c)$ colour gauge symmetry of course has to be gauge-fixed in the standard manner, including also Yang-Mills ghosts. For the moment, we simply include these Yang-Mills gauge-fixing terms implicitly in the gluon measure $\mathcal{D}\mu[G]$.

There is nothing unphysical implied by the coupling to external vector and axial vector sources; these sources only serve to define appropriate Green functions through functional differentiation. They are clearly not intrinsically part of QCD, and will eventually be set equal to zero. Nevertheless, they turn out to play a rather profound rôle in the derivation of the effective Lagrangian. They would also, of course, acquire a physical meaning if they were to include the couplings of the electroweak interactions. The γ -matrices are Hermitean and obey the usual Clifford algebra.

We next perform a field-enlarging transformation in which the fermion fields are chirally rotated by a local (abelian) field $\theta(x)$. It is essential that the field transformation is made only in a *regularized* version of the QCD generating functional. A convenient consistent scheme in the fermion sector is provided by a set of Pauli-Villars regulator fields. These regulators of course *only* regularize the fermionic sector of QCD. We still need to regularize also the gluon sector of QCD in order to interpret the resulting field-transformed Lagrangian as an effective Lagrangian with an ultraviolet cut-off Λ .

After the field transformation we will get two pieces, one classical from the variation of the action in eq.(2) under the local chiral rotation, and one quantum mechanical from the change of the fermionic measure [13]. We rearrange it into an expansion in decreasing powers of Λ , the ultraviolet cut-off:

$$\begin{aligned}\mathcal{Z}_\Lambda[V, A] &= \int \mathcal{D}_\Lambda[\bar{\chi}, \chi] \mathcal{D}\mu[G] e^{-\int d^4x \mathcal{L}'} \\ \mathcal{L}' &= \frac{1}{4g^2} \text{tr} G_{\mu\nu} G_{\mu\nu} + \bar{\chi} (\not{\partial} - i\not{A} - i\not{M} - i\not{A}\gamma_5 + i\not{\partial}\theta\gamma_5) \chi + \mathcal{L}_{WZ} + \mathcal{L}_J\end{aligned}\quad (3)$$

where the last two terms arise from the Jacobian of the transformation. The first part, the Wess-Zumino term, starts at $\mathcal{O}(\Lambda^0)$:

$$\mathcal{L}_{WZ} = -\frac{iN_f}{16\pi^2} \theta \epsilon_{\mu\nu\rho\sigma} \left(\text{tr} G_{\mu\nu} G_{\rho\sigma} - 4N_c \partial_\mu V_\nu \partial_\rho V_\sigma - \frac{4N_c}{3} \partial_\mu A_\nu \partial_\rho A_\sigma \right) + \mathcal{O}(\Lambda^{-2}), \quad (4)$$

while the second part reads $\mathcal{L}_J = \Lambda^2 \mathcal{L}_2 + \mathcal{L}_0 + \frac{1}{\Lambda^2} \mathcal{L}_{-2} + \frac{1}{\Lambda^4} \mathcal{L}_{-4} + \dots$, with

$$\begin{aligned}\mathcal{L}_2 &= \frac{N_f N_c \kappa_2}{4\pi^2} \left(A_\mu A_\mu - (A_\mu - \partial_\mu \theta)(A_\mu - \partial_\mu \theta) \right) \\ \mathcal{L}_0 &= \frac{N_f N_c}{24\pi^2} \left(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu (A_\nu - \partial_\nu \theta) \partial_\mu (A_\nu - \partial_\nu \theta) \right. \\ &\quad \left. + 2(A_\mu A_\mu)^2 - 2((A_\mu - \partial_\mu \theta)(A_\mu - \partial_\mu \theta))^2 \right) \\ \mathcal{L}_{-2} &= \frac{N_f \kappa_{-2}}{48\pi^2} \left(N_c \partial^2 A_\mu \partial^2 A_\mu - N_c \partial^2 (A_\mu - \partial_\mu \theta) \partial^2 (A_\mu - \partial_\mu \theta) \right. \\ &\quad \left. + (A_\mu A_\mu - (A_\mu - \partial_\mu \theta)(A_\mu - \partial_\mu \theta)) \text{tr}_c G_{\nu\rho} G_{\nu\rho} + \mathcal{O}(A_\mu^4) \right)\end{aligned}\quad (5)$$

We list only the first three terms; the whole expansion in increasing powers of inverse cut-off can be computed following the technique described in ref. [13]. The coefficients κ_2 in eq. (5) are regularization-scheme dependent constants [8].

When we next integrate over the collective fields in the path integral, a chiral gauge symmetry appears [5]:

$$\begin{aligned}\chi(x) &\rightarrow e^{i\alpha(x)\gamma_5}\chi(x) \\ \bar{\chi}(x) &\rightarrow \bar{\chi}(x)e^{i\alpha(x)\gamma_5} \\ \theta(x) &\rightarrow \theta(x) - \alpha(x).\end{aligned}\tag{6}$$

The transformed action (3) already contains a kinetic energy piece for $\theta(x)$:

$$\begin{aligned}\mathcal{L}' &= \frac{1}{4g^2} \text{tr} G_{\mu\nu} G_{\mu\nu} + \bar{\chi}(\not{\partial} - i\not{G} - i\not{M} - i\not{A}\gamma_5 + i\not{\partial}\theta\gamma_5)\chi \\ &+ \frac{N_f}{2} \partial_\mu \theta f^2 \partial_\mu \theta - N_f A_\mu f^2 \partial_\mu \theta - \frac{N_f N_c}{12\pi^2} \left(\left((A_\mu - \partial_\mu \theta)(A_\mu - \partial_\mu \theta) \right)^2 - (A_\mu A_\mu)^2 \right) \\ &- \theta \frac{iN_f}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} \left(\text{tr}_c G_{\mu\nu} G_{\rho\sigma} - 4N_c \partial_\mu V_\nu \partial_\rho V_\sigma - \frac{4N_c}{3} \partial_\mu A_\nu \partial_\rho A_\sigma \right) + \mathcal{O}(\Lambda^{-2}),\end{aligned}\tag{7}$$

where f^2 is an operator:

$$f^2 = -\frac{N_c \kappa_2 \Lambda^2}{2\pi^2} + \frac{N_c}{12\pi^2} \partial^2 - \frac{N_c \kappa_{-2}}{24\pi^2 \Lambda^2} \partial^2 \partial^2 - \frac{\kappa_{-2}}{24\pi^2 \Lambda^2} \text{tr}_c G_{\nu\rho} G_{\nu\rho} + \dots\tag{8}$$

The dots denote higher-order gluonic terms, derivatives and combinations of them divided by suitable powers of Λ . The implied expansion in inverse powers of the ultraviolet cut-off Λ is somewhat deceptive, since these powers are often multiplying operators of increasing dimension. This can lead to compensating factors of Λ in the numerators, thus putting the validity of the expansion in jeopardy. The only genuine expansion parameter will then be (low) momentum, or inverse powers of a large number of colours N_c .

Suppose we introduce a pseudoscalar field η_0 with the canonical dimension of *mass* as $\theta = \eta_0/\sqrt{N_f f_0}$, and view f_0 as a bare coupling. Identifying f_0^2 with the leading term in (8), $f_0^2 = -N_c \kappa_2 \Lambda^2/2\pi^2$, leads to a Lagrangian for a pseudoscalar field:

$$\mathcal{L}_{\eta_0} = \frac{1}{2} \partial_\mu \eta_0 \partial_\mu \eta_0 - A_\mu \sqrt{N_f f_0} \partial_\mu \eta_0 + \dots\tag{9}$$

The dots denote higher derivative terms, gluonic terms and self interactions of the pseudoscalar field. Note that f_0 is both cut-off dependent and scheme dependent. Let us now look at the higher derivative terms. In a perturbative sense, the propagator for the field η_0 can be derived from the bilinear part of \mathcal{L}' . It is a higher-derivative (or essentially Pauli-Villars) *regularized* bosonic propagator with a regulator mass proportional to Λ^2 . There is probably a simple reason for this: we are throughout performing field transformations within a *regularized* fermionic path integral. Even after a series of field-enlarging transformations (and the required gauge fixing of the new local symmetry) of such a form that we end up with new propagating fields, the generating functional is still ultraviolet regularized.

As explained above, useful forms of the effective Lagrangian are derived by judicious choices of gauge fixing. Whereas the gauge $\theta(x) = 0$ trivially gives us back cut-off QCD in its original formulation, almost all other gauge choices that remove part of the QCD degrees of freedom will lead to non-trivial effective Lagrangians. We base the gauge fixing it on the axial singlet current as given by a functional derivative with respect to A_μ at $A_\mu = 0$:

$$i\langle J_\mu^5 \rangle = i\langle \bar{\psi} \gamma_\mu \gamma_5 \psi \rangle = i\langle \bar{\chi} \gamma_\mu \gamma_5 \chi \rangle + N_f f^2 \partial_\mu \theta + \dots\tag{10}$$

The additional terms represented by dots are at least of third order in $\theta(x)$. The whole expression is of course gauge invariant, but the individual components on the r.h.s. are not. We now choose a gauge-fixing function Φ by

$$\Phi = i \frac{\partial_\mu}{N_f f_0^2 \partial^2} \bar{\chi} \gamma_\mu \gamma_5 \chi . \quad (11)$$

When we implement Φ as a delta-function constraint in the path integral, we must be careful that it has correct transformation properties. Under a global chiral rotation, $\bar{\chi}(x) \rightarrow \bar{\chi}(x) e^{i\alpha\gamma_5}$, $\chi(x) \rightarrow e^{i\alpha\gamma_5} \chi(x)$, $\bar{\chi} \gamma_\mu \gamma_5 \chi$ remains *classically* invariant but quantum mechanically it shifts due to the chiral anomaly: eq. (7):

$$i \frac{\partial_\mu}{N_f f_0^2 \partial^2} \bar{\chi} \gamma_\mu \gamma_5 \chi \rightarrow i \frac{\partial_\mu}{N_f f_0^2 \partial^2} \bar{\chi} \gamma_\mu \gamma_5 \chi + \alpha . \quad (12)$$

The action does not remain invariant either, but shifts due to the axial anomaly:

$$S' = \int d^4x \mathcal{L}' \rightarrow S' - 2iN_f \alpha \int d^4x \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{tr}_c G_{\mu\nu} G_{\rho\sigma} . \quad (13)$$

Assuming that we sum only over integer winding numbers, the action does, however, remain invariant under constant chiral rotations of the form $\alpha = n\pi/N_f$. This means that also $\theta(x)$ is only globally defined modulo π/N_f . The gauge-fixing constraint must respect the above periodicity property; there must, even in the gauge-fixed path integral, be no distinction between $\theta(x)$ and $\theta(x) + n\pi/N_f$. If we choose a δ -function constraint to implement the gauge choice, this δ -function must then necessarily be *globally periodic*. Representing it by an auxiliary field $b(x)$, the gauge-fixing function then provides a few new terms in the action. But they will in general modify the relevant chiral Jacobian, and for consistency new terms must be added to the action to compensate for this. This leads to a rather involved procedure of gauge fixing, and we give here a simpler derivation. We can consider the above gauge-fixing function as derived from a constraint in the *original* representation of QCD of the form

$$\Phi' = i \frac{\partial_\mu}{N_f f_0^2 \partial^2} \bar{\psi} \gamma_\mu \gamma_5 \psi + \theta . \quad (14)$$

We then introduce the δ -function constraint as

$$\delta(\Phi') = \int \mathcal{D}[b] \exp \left[- \int d^4x \left(\frac{i}{N_f f_0^2} B_\mu(x) J_\mu^5(x) + \theta(x) \partial_\mu B_\mu(x) \right) \right] , \quad (15)$$

where the axial vector field $B_\mu(x)$ is defined by

$$B_\mu(x) \equiv \int d^4y d^4z b(y) \partial_{(y-z)}^{-2} \partial_\mu^{(z)} \delta(z-x) , \quad (16)$$

i.e., $b(x) = -\partial_\mu B_\mu(x)$. Global periodicity of the δ -function means that b is constrained,

$$\int d^4x b(x) = - \int d^4x \partial_\mu B_\mu = ikN_f , \quad (17)$$

where k is an arbitrary integer. This global constraint means that $b(x)$ (or $\partial_\mu B_\mu$) share certain properties with topologically non-trivial fields.

After gauge-fixing the only remnant of the $U(1)$ axial gauge symmetry is the BRST symmetry

$$\begin{aligned}\delta\bar{\chi}(x) &= i\bar{\chi}(x)\gamma_5 c(x) \\ \delta\chi(x) &= -ic(x)\gamma_5\chi(x) \\ \delta\theta(x) &= -c(x) \\ \delta\bar{c}(x) &= b(x),\end{aligned}\tag{18}$$

and $\delta c(x) = \delta b(x) = 0$. The ghost term is trivial in the present case, being just $\bar{c}c$.

It is convenient to choose a slightly different gauge which will only affect higher order correlation functions of our gauge fixing expression Φ' . We simply choose to add BRST-invariant terms involving powers of $b(x)$ such that the gauge fixed Lagrangian looks like

$$\begin{aligned}\mathcal{L}'' &= \bar{\chi}\left(\not{\partial} - i\not{G} - i\not{M} - i\left(A - \frac{1}{N_f f_0^2}\not{B} - \not{\partial}\theta\right)\gamma_5\right)\chi + \bar{c}c + \mathcal{L}_{YM} \\ &+ \frac{N_f f_0^2}{2}\partial_\mu\theta\partial_\mu\theta - N_f f_0^2 A_\mu\partial_\mu\theta \\ &- \frac{N_f N_c}{12\pi^2}\left(\left((A_\mu - \frac{1}{N_f f_0^2}B_\mu - \partial_\mu\theta)(A_\mu - \frac{1}{N_f f_0^2}B_\mu - \partial_\mu\theta)\right)^2 - (A_\mu A_\mu)^2\right) \\ &- \theta\frac{iN_f}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}\left(\text{tr}_c G_{\mu\nu}G_{\rho\sigma} - 4N_c\partial_\mu V_\nu\partial_\rho V_\sigma - \frac{4N_c}{3}\partial_\mu A_\nu\partial_\rho A_\sigma\right) + \mathcal{O}(\Lambda^{-2}).\end{aligned}\tag{19}$$

In order to view (19) as an *effective Lagrangian*, we need additional input. The obvious choice would be to identify the θ -field with the flavour-singlet pseudoscalar field of the η' meson, in appropriate units. Certainly, eq. (19) gives the correct QCD action for describing the low-momentum dynamics of the composite operator $J_\mu^5(x) = i\bar{\psi}\gamma_\mu\gamma_5\psi(x)$ of the original quark fields. Taking one partial derivative, we can equally well describe $\partial_\mu J_\mu^5(x)$, which is a non-zero operator due to the chiral anomaly. It has quantum numbers $J^{PC} = 0^{-+}$, and is a singlet under flavour. As such, it should have a non-vanishing overlap with the physical η' meson. For example, if we were able to compute the long-distance fall-off of the corresponding two-point correlation function, this should provide us with the mass of the lowest-lying state of these quantum numbers. By definition, this is the mass of the η' meson.

Going back to eq. (2), we note that the connected 2-point function of $\partial_\mu J_\mu^5(x)$ can be obtained by differentiating twice with respect to a pseudoscalar source $\sigma(x)$ defined by splitting $A_\mu = \partial_\mu\sigma(x) + A_\mu^T$ into a longitudinal and a transverse part. Shifting B_μ

$$B_\mu(x) \rightarrow B_\mu(x) + N_f f_0^2 \partial_\mu\theta(x) - N_f f_0^2 \partial_\mu\sigma(x)\tag{20}$$

leads to a Lagrangian

$$\begin{aligned}\mathcal{L}''' &= \bar{\chi}\left(\not{\partial} - i\not{G} - i\left(A^T - \frac{1}{N_f f_0^2}\not{B}\right)\gamma_5\right)\chi + \mathcal{L}_{ghost} + \mathcal{L}_{YM} \\ &- \frac{N_f N_c}{12\pi^2}\left(\left((A_\mu^T - \frac{1}{N_f f_0^2}B_\mu)(A_\mu^T - \frac{1}{N_f f_0^2}B_\mu)\right)^2 - (A_\mu A_\mu)^2\right) \\ &+ \frac{N_f f_0^2}{2}\partial_\mu\theta\partial_\mu\theta - N_f f_0^2 \partial_\mu\sigma\partial_\mu\theta \\ &- \theta\frac{iN_f}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}\left(\text{tr}_c G_{\mu\nu}G_{\rho\sigma} - 4N_c\partial_\mu V_\nu\partial_\rho V_\sigma - \frac{4N_c}{3}\partial_\mu A_\nu^T\partial_\rho A_\sigma^T\right) + \mathcal{O}(\Lambda^{-2})\end{aligned}\tag{21}$$

Apart from contact terms only a linear coupling of σ to θ is left. The remaining part of $\mathcal{O}(\Lambda^{-2})$ is also independent of θ because it contains θ only in the combination $B_\mu + N_f f_0^2 \partial_\mu \theta$; after the shift (20) θ disappears from these terms.

We can now derive some exact Ward identities, setting the external sources to zero: The original anomalous Ward identity

$$\partial_\mu \langle J_{5\mu} \rangle = i \partial_\mu \langle \bar{\psi} \gamma_\mu \gamma_5 \psi \rangle = -i \frac{N_f}{16\pi^2} \langle G\tilde{G} \rangle + \mathcal{O}(\Lambda^{-2}) \quad (22)$$

is now just the equation of motion for the field θ :

$$f_0^2 \partial^2 \langle \theta \rangle = -i \frac{1}{16\pi^2} \langle G\tilde{G} \rangle + \mathcal{O}(\Lambda^{-2}) . \quad (23)$$

Analogously, we find for the 2-point function in the original QCD representation:

$$\langle \partial_\mu J_{5\mu}(x) \partial_\nu J_{5\nu}(y) \rangle = - \left(\frac{N_f}{16\pi^2} \right)^2 \langle G\tilde{G}(x) G\tilde{G}(y) \rangle - N_f f_0^2 \partial^2 \delta(x-y) + \mathcal{O}(\Lambda^{-2}) . \quad (24)$$

The same identity can be derived from a simple infinitesimal shift of θ to second order:

$$N_f^2 f_0^4 \langle \partial^2 \theta(x) \partial^2 \theta(y) \rangle = - \left(\frac{N_f}{16\pi^2} \right)^2 \langle G\tilde{G}(x) G\tilde{G}(y) \rangle - N_f f_0^2 \partial^2 \delta(x-y) + \mathcal{O}(\Lambda^{-2}) \quad (25)$$

This illustrates that for relevant Green functions our gauge identifies as *operators*

$$\partial_\mu J_{5\mu} \sim N_f f_0^2 \partial^2 \theta . \quad (26)$$

The gauge-fixing procedure presented above thus amounts to introducing explicitly, at the Lagrangian level, an “interpolating” field according to the relation (26).

Just in order to illustrate how non-trivial results can be extracted from the effective Lagrangian (21), let us consider a very crude approximation, the cumulant expansion. In this framework we can integrate out all fields in (21) except θ to arrive at an effective Lagrangian

$$\mathcal{L}_{eff} = \frac{F_0^2}{2} \partial_\mu \theta \partial_\mu \theta + \frac{F_0^2 M_0^2}{2} \theta^2 + \dots \quad (27)$$

The dots denote higher derivative terms and self-interactions of order θ^3 . The parameters F_0 and M_0 are defined through

$$F_0^2 M_0^2 = \int d^4x \left\langle \frac{N_f}{16\pi^2} G\tilde{G}(x) \frac{N_f}{16\pi^2} G\tilde{G}(0) \right\rangle_{trunc} \quad (28)$$

and

$$F_0^2 = N_f f_0^2 - \int d^4x \frac{x^2}{8} \left\langle \frac{N_f}{16\pi^2} G\tilde{G}(x) \frac{N_f}{16\pi^2} G\tilde{G}(0) \right\rangle_{trunc} . \quad (29)$$

The expression for $F_0^2 M_0^2$ in (28) looks similar to the one derived by Witten [14] and Veneziano [15] in the limit $N_c \rightarrow \infty$. But the expectation values $\langle \dots \rangle_{trunc}$ have to be taken with respect to a “truncated” version of QCD:

$$\mathcal{L}_{trunc} = \bar{\chi} \left(\not{\partial} - i \not{A} + i \frac{1}{N_f f_0^2} \not{B} \gamma_5 \right) \chi + \mathcal{L}_{ghost} + \mathcal{L}_{YM} - \frac{N_c}{12\pi^2 N_f^3 f_0^8} \left(B_\mu B_\mu \right)^2 + \mathcal{O}(\Lambda^{-2}). \quad (30)$$

At first glance one would argue that the topological susceptibility has to be zero in such a theory because of the massless quarks. If this were true, then our field θ would be massless. Indeed, for the full theory we can derive from the Ward identities (25) that the r.h.s. of eq. (28) vanishes. This is a well-known result in massless QCD. But if we make an analogous step in the *truncated* theory, we get instead

$$\int d^4x \langle \partial_\mu B_\mu(x) \partial_\nu B_\nu(0) \rangle_{trunc} = -\left(\frac{N_f}{16\pi^2}\right)^2 \int d^4x \langle G\tilde{G}(x)G\tilde{G}(0) \rangle_{trunc} + \mathcal{O}(\Lambda^{-2}) \quad (31)$$

It is the constraint (17) that suggests a non-vanishing topological susceptibility in the truncated theory, and thus a non-zero mass for the field θ . Without this constraint, one could decouple the B_μ -field from the fermions through a chiral rotation [16], and the standard proof of a vanishing topological susceptibility in the theory with massless quarks would go through.

Witten has argued [14] that the vanishing of the topological susceptibility in massless QCD can be considered as a cancellation of the pure gluonic part by a contribution from the η' meson. From our point of view, the possibility of a non-zero topological susceptibility in the *truncated* QCD arises due to the gauge constraint, which precisely can be thought of as removing the η' -part of the topological susceptibility.

One obvious difficulty with the present approach is that everything is expressed in terms of bare parameters in the cut-off theory. Being explicitly cut-off dependent, it is also scheme dependent. The whole set of effective one-loop interactions between the bosonic collective fields and left-over QCD degrees of freedom indeed follow directly from the Pauli-Villars regulator fields. This is just as in the solvable case of two dimensions [6]. As it stands, the unrenormalized theory has, with massless quarks, only one mass scale: that of the cut-off Λ . This means that all dimensionful couplings in the effective theory are given by powers of this ultraviolet cut-off. In the renormalized theory this cut-off becomes replaced by a physical mass scale, essentially Λ_{QCD} . In the end, if one integrates out all gluonic and quark degrees of freedom and leaves only the collective fields, the physical couplings are directly related to gluonic and fermionic correlators, moments thereof, and condensates. The precise relationship between the couplings of the collective field Lagrangian and these vacuum expectation values will be of roughly the kind discussed in the case of the Witten-Veneziano relation above. The fact that physical couplings will be related to these Green functions is also evident if we return to the definition of the gauge-fixing condition (11). The term containing f_0^2 should in fact depend on f^2 with contributions also from gluonic fields, and a full treatment should incorporate the effect of integrating out the gluonic degrees of freedom. Intuitively, one expects that one major effect of such a renormalization program is to relate f^2 to gluonic condensates.

Finally, we shall briefly outline the generalization of the present effective Lagrangian technique to the case of the $SU(N_f)$ pseudoscalar multiplet. In contrast with the $U(1)$ case discussed above, the flavour non-singlet axial currents are exactly conserved in the limit of massless quarks. How do we introduce the appropriate collective fields for this non-Abelian (flavoured) case? As before, the main input is the choice of quantum numbers we wish to describe. We then start again with a generating functional of QCD, eq.(2), where we now take external sources V_μ and A_μ to be elements of $SU(N_f)$. It is now most

convenient to introduce collective fields $\theta(x)$ by, *e.g.*, purely left-handed transformations:

$$q_L(x) = e^{2i\theta(x)} \chi_L(x) , \quad \bar{q}_L(x) = \bar{\chi}_L(x) e^{-2i\theta(x)} \quad (32)$$

i.e. local phase transformations acting only on the left-handed spinors, $q_L = P_+ q$, $\bar{q}_L = \bar{q} P_-$, $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$. Also, define $L_\mu = V_\mu + A_\mu$, $R_\mu = V_\mu - A_\mu$.

The transformation (32) causes a change of the regularized fermionic functional integral measure due to its handedness. In order to calculate the corresponding contribution to the Lagrangian, we again use Pauli-Villars regularization:

$$\begin{aligned} \mathcal{Z}_\Lambda[V, A] &= \int \mathcal{D}_\Lambda[\bar{\chi}, \chi] d\mu[G] e^{-\int d^4x \mathcal{L}'} \\ \mathcal{L}' &= \bar{\chi} \gamma_\mu (\partial_\mu - iG_\mu - iL_\mu^\theta P_+ - iR_\mu P_-) \chi + \mathcal{L}_J + \mathcal{L}_{WZ} + \mathcal{L}_{YM} , \end{aligned} \quad (33)$$

where, with $U(x) = e^{2i\theta(x)}$, L_μ is modified to $L_\mu^\theta = U^\dagger L_\mu U + iU^\dagger \partial_\mu U$.

The positive parity part can, as in the abelian case, be ordered as an expansion in inverse powers of the ultraviolet cut-off Λ . The first three terms are given by

$$\begin{aligned} \mathcal{L}_2 &= \frac{N_c \kappa_2}{4\pi^2} \text{tr}_f A_\mu^{(s)} A_\mu^{(s)} \Big|_{s=1}^0 \\ \mathcal{L}_0 &= \frac{N_c}{8\pi^2} \text{tr}_f \left(-i F_{\mu\nu}^{(s)} [A_\mu^{(s)}, A_\nu^{(s)}] + \frac{1}{3} D_\mu^{(s)} A_\nu^{(s)} D_\mu^{(s)} A_\nu^{(s)} - \frac{2}{3} (A_\mu^{(s)} A_\mu^{(s)})^2 \right. \\ &\quad \left. + \frac{4}{3} A_\mu^{(s)} A_\nu^{(s)} A_\mu^{(s)} A_\nu^{(s)} \right) \Big|_{s=1}^0 \\ \mathcal{L}_{-2} &= \frac{\kappa_{-2}}{48\pi^2} \text{tr}_f \left(N_c \partial^2 A_\mu^{(s)} \partial^2 A_\mu^{(s)} + A_\mu^{(s)} A_\mu^{(s)} \text{tr}_c G_{\nu\rho} G_{\nu\rho} + \dots \right) \Big|_{s=1}^0 . \end{aligned} \quad (34)$$

The terms omitted in \mathcal{L}_{-2} and denoted by dots are at least fourth order in $A_\mu^{(s)}$ and $V_\mu^{(s)}$.

The leading term of the negative parity part is the integrated Bardeen-anomaly:

$$\begin{aligned} \mathcal{L}_{WZ} &= \frac{i}{16\pi^2} \int_1^0 ds \epsilon_{\mu\nu\rho\sigma} \text{tr}_f \text{tr}_c \theta \left(F_{\mu\nu}^{(s)} F_{\rho\sigma}^{(s)} + \frac{1}{3} A_{\mu\nu}^{(s)} A_{\rho\sigma}^{(s)} + \frac{32}{3} A_\mu^{(s)} A_\nu^{(s)} A_\rho^{(s)} A_\sigma^{(s)} \right. \\ &\quad \left. + \frac{8i}{3} (F_{\mu\nu}^{(s)} A_\rho^{(s)} A_\sigma^{(s)} + A_\mu^{(s)} F_{\nu\rho}^{(s)} A_\sigma^{(s)} + A_\mu^{(s)} A_\nu^{(s)} F_{\rho\sigma}^{(s)}) \right) + \mathcal{O}(\Lambda^{-2}) . \end{aligned} \quad (35)$$

The covariant derivatives and field strength tensors appearing in eqs. (34) and (35) are defined as $\mathcal{D}_\mu A_\nu = \partial_\mu A_\nu - i[V_\mu, A_\nu]$, $A_{\mu\nu} = \mathcal{D}_{[\mu} A_{\nu]}$, $F_{\mu\nu} = \partial_{[\mu} V_{\nu]} - i[V_\mu, V_\nu] - i[A_\mu, A_\nu]$, and the transformed fields appearing in (34) and (35) as

$$\begin{aligned} V_\mu^{(s)} &= \frac{1}{2} \left(e^{-2is\theta} L_\mu e^{2is\theta} + i e^{-2is\theta} \partial_\mu e^{2is\theta} + R_\mu \right) \\ A_\mu^{(s)} &= \frac{1}{2} \left(e^{-2is\theta} L_\mu e^{2is\theta} + i e^{-2is\theta} \partial_\mu e^{2is\theta} - R_\mu \right) . \end{aligned} \quad (36)$$

The parameter s ranging from 0 to 1 thus defines a continuous transformation which, for $s = 1$, coincides with (32).

When we now integrate over the invariant Haar measure $\int \mathcal{D}[U]$, a new local non-Abelian gauge symmetry emerges:

$$\begin{aligned} \chi_L(x) &\rightarrow e^{2i\alpha(x)} \chi_L(x) \\ \bar{\chi}_L(x) &\rightarrow \bar{\chi}_L(x) e^{-2i\alpha(x)} \\ U(x) &\rightarrow U(x) e^{-2i\alpha(x)} . \end{aligned} \quad (37)$$

As in the flavour-singlet case, the crucial next step is the choice of gauge fixing. While we have a lot of freedom available in choosing the gauge-fixing function, it is not immediately obvious what will lead to useful representations. In that respect the abelian case is far better under control, since we there can rely on experience gained in the solvable two-dimensional case. To achieve the same amount of simplification in this non-Abelian case seems to require that we know how to perform the “smooth” analogue [6] of non-Abelian bosonization. Still, even without explicitly specifying the non-Abelian gauge-fixing function, it is clear that the final result will be closely related to what has become known as the constituent chiral quark model of Manohar and Georgi [17]. This is a chiral Lagrangian coupled to remnant quark degrees of freedom (in our language the χ -fields) and the gluons. These couplings are, apart from those induced by the gauge fixing, entirely specified by QCD in our approach. But to really analyze the resulting effective Lagrangian requires that we decide on a useful gauge-fixing function. Work in that direction is still in progress.

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